How Blending Illuminates Understandings of Calculus

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Abstract

Conceptual blending is gaining momentum amongst mathematics educators interested in better conceptualizing mathematical meanings students are building. We used conceptual blending as a lens to illuminate students' understandings of calculus concepts as they emerged during sustained mathematical inquiry. We share some of the insights we have gained by using this lens in our analysis. Viewing the mathematical connections along with the emergent structure that follows allowed us to more fully characterize students' constructions of meaning for mathematics. Additionally we have found that conceptual blending is flexible in the unit of analysis, aids comparisons between conceptions held by a student or different students, brings to the forefront elements of the input and blended spaces and the connections between them, emphasizes the meaning that students are building for important mathematics.

Conceptual blending (Fauconnier & Turner, 2002) is gaining momentum amongst mathematics educators interested in better conceptualizing mathematical meanings students are building (Núñez, 2004; Megowen & Zandieh, 2005). We use conceptual blending as a lens to illuminate individual and collective understandings of calculus concepts as they emerge from sustained mathematical inquiry. In this paper, we share some of the insights we have gained through using conceptual blending in our analysis.

Theoretical perspective

Agency and Purposeful Choice

We defined personal Agency as "the requirement, responsibility and freedom to choose based on prior experiences and imagination, with concern not only for one's own understandings of mathematics, but with mindful awareness of the impact one's actions and choices may have on others" (Walter & Gerson, 2007 p. 209). We further suggest that agency is a requirement for leaning to occur. Because we hold this view we carefully examine students choices and the connected understanding they build through this lens.

Conceptual Blending

Fauconnier and Turner (2002) elucidate the construction of meaning through their cognitive theory of conceptual blending. They suggest that human beings create new meaning by combining two input spaces to form a blended space making new relations available in the blend that are not present in the input spaces (Figure 1). We thereby bring together previous knowledge (e.g. past experiences, prototypical examples, language, context, frames, and scripts, etc.) into a creative and imaginative combination, or blend, with its own emergent structure resulting in "genuine novel integrated action" (p. 35).

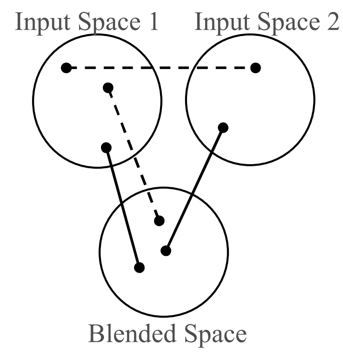


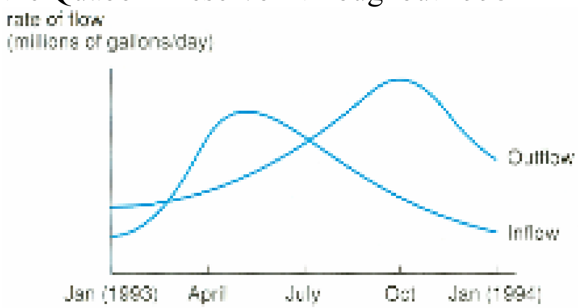
Figure 1: Conceptual blend map

The emergent structure, or the meaning of a blend, is created in three ways: composition, completion, and running the blend. Composition means combining elements to create relations that do not exist in the separate inputs. Completion, involves running a frame, from one of the input spaces, in the blended space thus building new creative connections in the blend. Running the blend is imagining a simulation of the blend and seeing what new insights emerge.

Setting

In winter, 2006, the authors team-taught and conducted a teaching experiment, in university honors calculus I, in which 22 students collaboratively explored cognitively important, multiple-response tasks. Tasks were designed by the authors or adapted from various sources to elicit important calculus content. Students were encouraged to develop multiple solution strategies and to justify their answers without prior instruction. Pedagogical decisions were based upon ideas that students brought forward. Students' activation of agency was recognized as necessary for the learning process (Walter & Gerson, 2007), and therefore, students' ideas were highly valued and pursued.

The Quabbin Reservoir in the western part of Massachusetts provides most of Boston's water. The graph below represents the flow of water in and out of the Quabbin Reservoir throughout 1993



(a) Sketch a possible graph for the quantity of water in the reservoir, as a function of time.
(b) Explain the changes in the quantity of water in the reservoir in terms of the relationships between outflow and inflow during each quarter of the year. How are these changes evident in your graph in part (a)?
(c) How does the quantity of water in the reservoir in Jan 1993 compare with the quantity of water in the reservoir in Jan 1994? How do you know?

Figure 2: The Quabbin Reservoir Task

Six weeks into the class students were given the Quabbin Reservoir Task (Figure 2). This task was adapted from a problem in the Harvard Consortium calculus (Hughes-Hallett et al., 1994). The researchers determined that this task would engage students with calculus concepts that they had not yet learned, provide a rich context, in which they would have some experience, and require high-level thinking. We suspected that important calculus content such as interpreting rates, the antiderivative, concavity, extrema, points of inflection, area between curves, and average rate of change would emerge from the discussion of the Quabbin Reservoir Task. However, we were intent upon allowing the students to bring forward the ideas, and build upon them to ultimately learn the calculus content. Thus the students' agency played a predominant role in establishing the content covered and the direction of inquiry.

Calculus content could be viewed as locations, or way points on a map set out by the learning outcomes for the course. The essential way points for a particular task were determined by the students during problem solving, and our role as instructors was to support student inquiry. We provided students with rich and engaging mathematical tasks that would allow them to forge their own paths to way points on the map, on each class day. Students visited and revisited important calculus content, but not necessarily along a path pre-determined by the instructors[i]. For example, suppose the concept of derivative was represented by a mountain on the map. At various times during the class, different students might view the mathematical terrain very differently. Some might see a mountain and choose to walk around the mountain to study it from below, climb the mountain, or view it from a helicopter. All of the students might navigate to way points on the mountain multiple times along different paths (and in the context of different tasks), but their individual experiences, because of their developmental journey may be quite varied. The mathematics tasks became anchors for common experiences upon which students could build calculus understandings and communicate with one another.

Brief Review of Relevant Research

Interpretation of Graphs and Emergent Understanding

Schnepp and Chazan (2004) suggest that interpreting graphs of motion, or rate of change, involves interpreting both the graph and the motions that the graph might describe. Students' interpretations and construction of graphs emerge over time and therefore need to be viewed and analyzed as the conceptions unfold (Roth & Lee, 2003). Collaborative learning in inquiry-based classrooms is emergent and dynamic and therefore must be viewed throughout the exploration, taking into account both individual performances as well as the performance of the collective (Martin, Towers, & Pirie; 2006). We suggest that both content, and connections students make amongst foundational calculus ideas such as derivative, antiderivative, and area between curves, as well as context and previous knowledge will give us a richer picture of the emergent meanings students are creating as they explore meaningful mathematics tasks. Our work builds upon the work of Núñez (2005) and Megowen & Zandieh (2005).

Research Questions

As a part of the larger study we were interested in studying how students working on the Quabbin Reservoir Task collaboratively built connected understanding of the quantity of water in the reservoir. In particular we are interested in studying their development over time of the fundamental theorem of calculus. For this paper we are interested in discussing how conceptual blending aided our analysis of connected understanding.

Research Methodology

Qualitative data were extracted from three hours of videotape collected within two class periods, in which four students explored the Quabbin Reservoir Task and presented their ideas to the class. Data were examined through a multilayer analysis of video, transcript, and original student work. Key episodes in which students were working with, or articulating understanding of, a conceptually important calculus idea, were identified, organized chronologically within each mathematical topic, and coded for content and connections.

In addition, single episodes and groups of episodes were analyzed by creating maps of conceptual blends. We used these maps as a representation of the students' connected understandings. While the students formed and articulated the blends, the maps were a creation of the researcher. These maps became part of the data being analyzed.

We were aware that inserting researcher created maps into the data could introduce bias or incorrect interpretations of students' learning. To minimize these problems the maps were triangulated amongst all data sources and were viewed as secondary sources rather than primary sources. That is, the video, student work, and field notes always took precedence over the researcher created maps. Existent and emergent theories were triangulated with all data sources resulting in a multilayered analysis supported by strong evidence. The maps were used to build and test theories and to help illuminate connected understandings.

Data and Analysis

Example 1: Miscommunication

Eight minutes into the class, two minutes after beginning the Quabbin Reservoir Task, Shaun showed his graph of the quantity of water in the reservoir to Timbre (Figure 3). Timbre laughed and said, "That's not what I was thinking." In response, Shaun explained his graph.

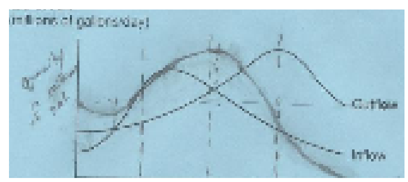


Figure 3: Shaun's graph of quantity

(00:08:42) **Shaun**: Because whenever the outflow is greater than the inflow the quantity is going down. Whenever they meet it levels out. And of course whenever the inflow is higher the quantity goes up.

Shaun's use of "going down" shows that he connected net flow with change in quantity.

Timbre and Jay, on the other hand, initially connected the net flow directly with quantity. Timbre first drew a graph of the net flow and labeled it Quantity. This resulted in a graph with a negative *y*-intercept [Figure 4]. Thus, when Shaun presented his graph, Timbre disagreed with his starting point.

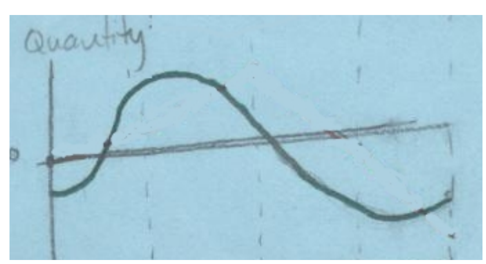


Figure 4: Timbre's initial graph of quantity

(00:09:05) **Timbre**: But your number is high.

(00:09:07) **Shaun**: Well we don't know where it starts at; we don't know how much is in there.

(00:09:11) **Timbre**: But we do know relatively going up, that's going to be more gallons, right, than going down. So if there is less [inflow], more water is going out, doesn't that mean that there would be less quantity?

Jay agreed with Timbre that the "first value [of the quantity] is a negative…because there is more outflow than inflow."

When confronted with Shaun's disagreement, Timbre and Jay worked independently, apparently convinced by Shaun's argument for a positive starting quantity. They both shifted their quantity graphs up, to reflect a positive starting quantity (Figure 5).

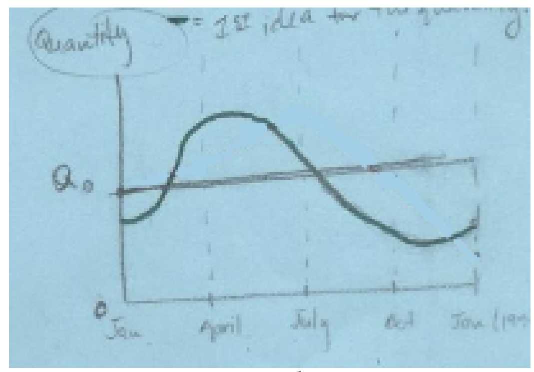


Figure 5: Timbre's 2nd graph of quantity

Shaun's, Timbre's, and Jay's language remained the same until 19:41 when Timbre began to compare her graph more closely with Shaun's. Thus for about the first 15 minutes of working on the task, Jay, Timbre and Shaun were communicating about their quantity graphs, at times thinking they were talking about the same thing, but were talking about two different graphs. Shaun was talking about the quantity graph and Timbre and Jay were talking about the net flow or the net flow shifted up.

Looking at the maps of the blends (Figures 6 and 7) the students were creating sheds light on the miscommunication and upon the ways the students were defining the quantity of water. In Shaun's conceptual blend for the quantity of water, he first creates a blend of the net flow by combining the inflow and outflow graphs (although he does not draw a graph of the net flow). We see that he is combining elements from the net flow or the combination of the inflow and outflow graphs, and the context of the reservoir. This allows him to blend together ideas such as: positive net flow, with rising water level to get increasing quantity. And from the context of the reservoir space he was able to bring down the idea that there is always water in the reservoir to the blended space where this information means that the quantity can't be negative.

Timbre and Jay used as their first input space the inflow and outflow space and like Shaun, blended with the context of the reservoir. When Timbre and Jay created their blend, they used virtually the same information that Shaun used. One difference was that Jay and Timbre blended the same ideas of positive net flow, with rising water level to get quantity rather than increased quantity. So Shaun's intermediate blend of net flow is created as quantity for Jay and Timbre. When they realize that there is always water in the reservoir, Jay and Timbre simply shift their "quantity" graph up to represent a positive starting quantity.

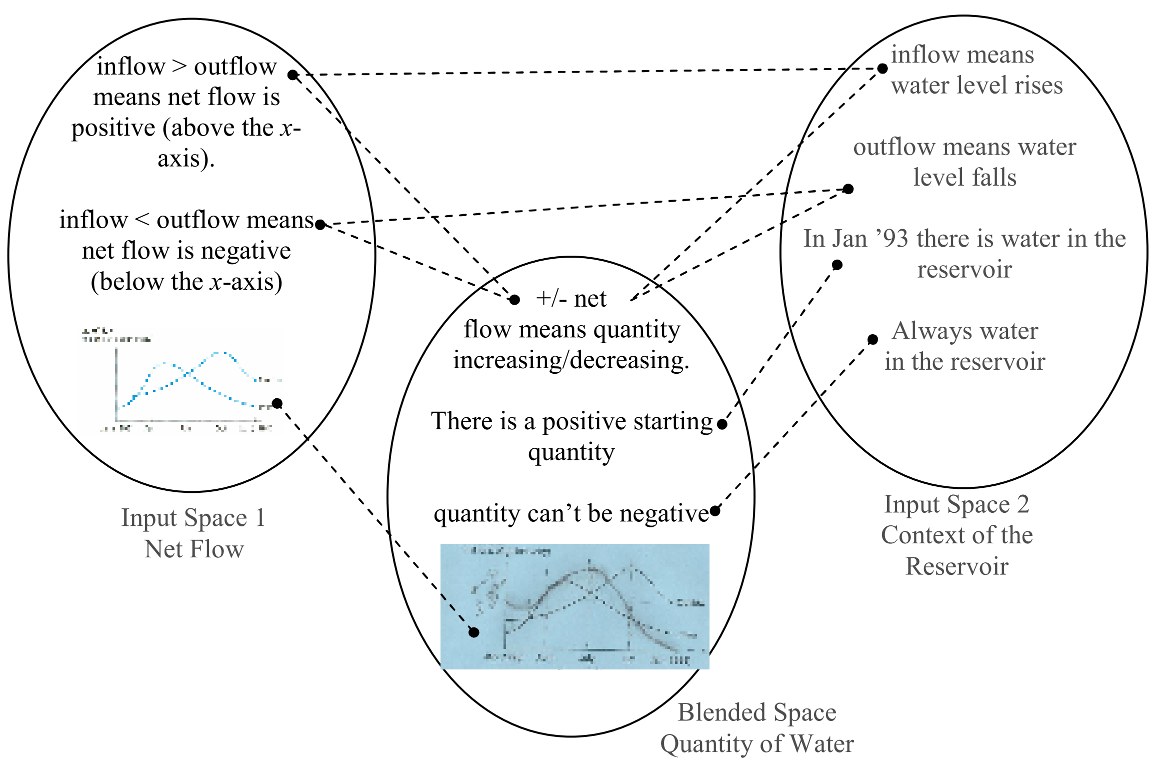


Figure 6: Shaun's blend of quantity in the first 15 minutes

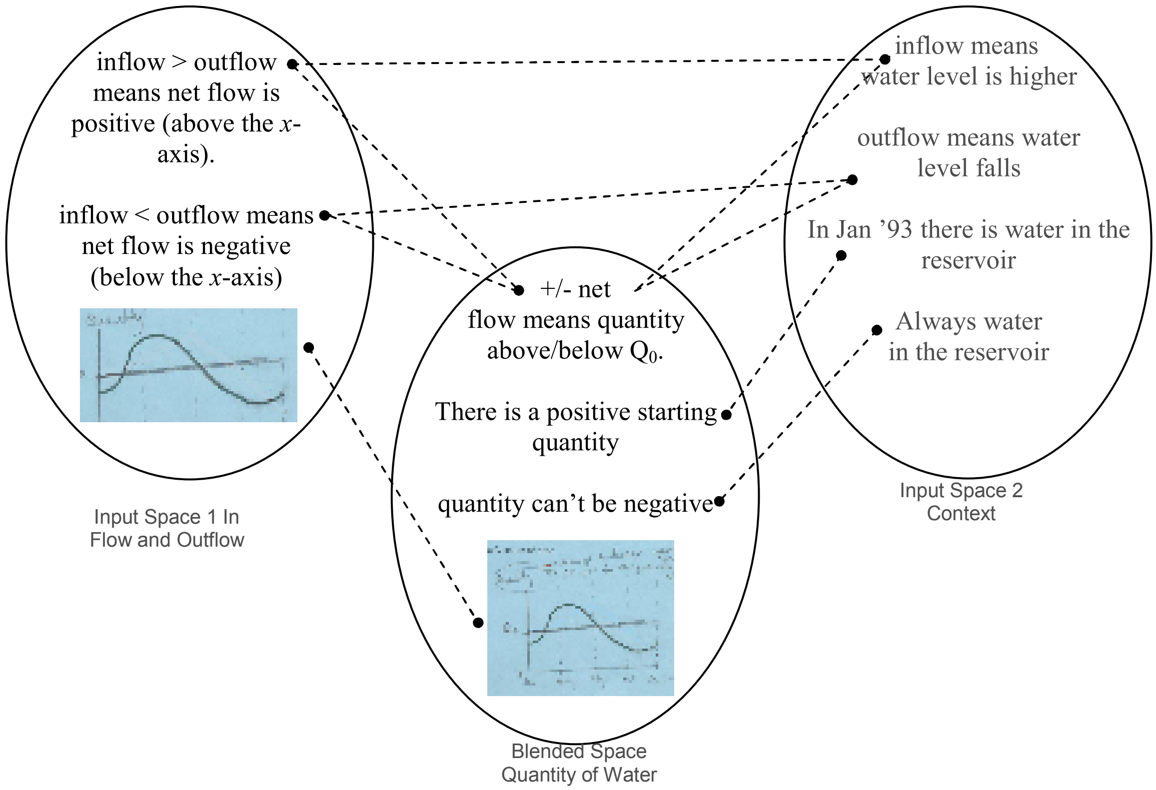


Figure 7: Timbre's and Jay's blend of quantity in the first 15 minutes

Maps of Conceptual Blends Aid Comparison

The maps for Shaun's and Timbre's and Jay's conceptual blends for the quantity of water in the reservoir had exactly the same structure. They were relying on the same blended ideas, but creating very different blended spaces. Without the blend we could see the differences in language meaning, negative vs. negative direction. The embodied difference in conception, below vs. decreasing (Lakoff and Núñez, 2000). The blend helped us to understand the differences in connected mathematical meaning. Timbre and Jay believed they were talking about the same thing as Shaun. This makes sense because they were using the same information to build their blends, and the structure of their blends was the same.

Example 2: Jay's Bowl Metaphor

After a lively discussion about how to compare the quantity of water in January 1993 and January 1994, about 17 minutes after beginning the task, Jay and Timbre began to make a new blend for the quantity of water by analyzing the area between the inflow and outflow curves. As a part of his new blend for using area between curves to determine the quantity of water, Jay introduced a metaphor of a bowl as a container for the area between curves.

(00:35:22) **Jay**: Okay, well look at this. Just look at that section with this section put together … because that's that right there <u>that little bowl</u>, that's all of the inflow, right? That's all <u>the gain in inflow</u>…And then that bowl, then that bowl are all the outflow, so I think you are right.

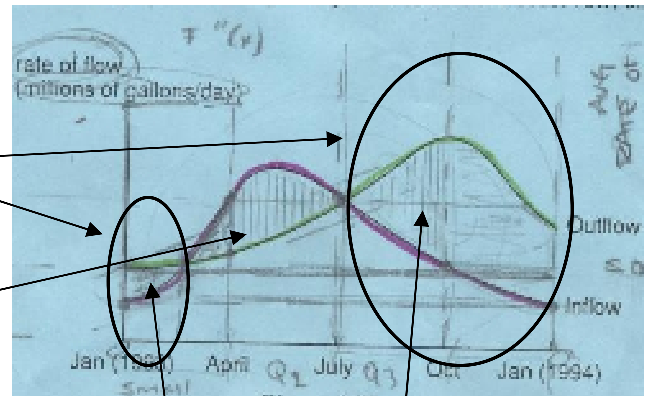


Figure 8: Jay's bowls

Analyzing Jay's speech allowed us to see that his conception of quantity was changing. For instance, in the previous example he equated inflow with a positive quantity of water in the reservoir, or a quantity above the starting quantity. Here he has changed to equating the area between curves with the "gain in inflow." We were able to see that Jay was using a metaphor, but some research team members did not understand the metaphor. So we created the map of the Jay's blend (Figure 9).

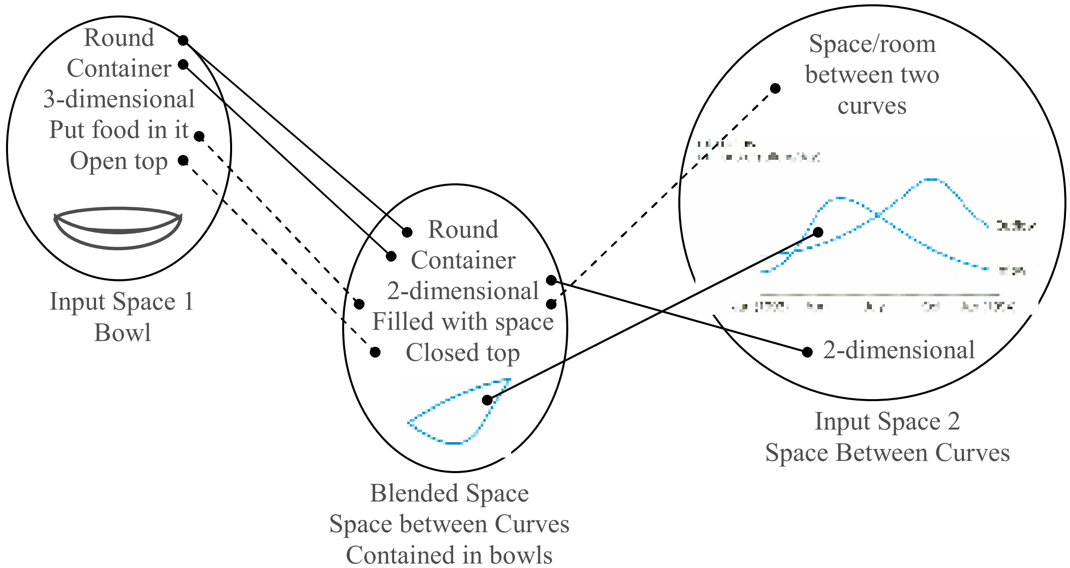


Figure 9: Map of Jay's Bowl Metaphor

Connections and Creativity

Mapping the blend helped us to see the tensions and the ground as connections between spaces and to understand better the meaning Jay was creating by blending a bowl with area between curves. In addition, mapping the blend helped to bring out the creativity that Jay was using to create meaningful mathematics.

Example 3: Running the blends

Once a map has been created by the researcher, she or he can run the blend to test theories or to compare the map of the blend with the discourse that follows. This allows the researcher to check for alignment between the data and interpretations of the data. Returning to Timbre's and Jay's blends of quantity in the first 15 minutes (Figure 7), recall that Timbre and Jay initially interpreted the quantity of water as the net flow of water into the reservoir shifted up to reflect a positive starting quantity. If we run Timbre's and Jay's blend, since net flow is determined discretely rather than globally in the given graphs, one would expect that when comparing the quantities of water at the beginning and end of the year, they would use a discrete comparison of the inflow and outflow in January 1993 with January 1994.

(00:22:16) **Jay**: … how does January ninety-four compare with the quantity in ninety-three? They look to be about the same inflow, but the outflow is really different

(00:22:50) **Shaun**: If you were to compare it, I think it would be better to take like the area of the outflow versus the area of the inflow and I think it's fairly similar but it does look like the outflow is going to be a little bit higher. Do you see what I'm saying?

(00:23:06) **Jay**: Oh you mean the total.

(00:23:08) **Shaun**: Yeah, if you were to consider like the area of the outflow versus the area of the inflow, then that would give you the overall total change to compare one year versus the other. Okay.

(00:23:19) **Timbre**: Well, It's not years it's months. Like at those points.

(00:23:20) **Shaun**: Well, but its from January to January so one year to the other

(00:23:24) **Timbre**: Well it says compare January 2003 [sic] with the quantity in [January 1994], so the way I see it is you're taking this and this, not with that whole space in between. It's not asking for the year 1993, it's asking for January 1993, so I see it, what's the quantity here, what's the quantity here, and how do they compare.

In fact, Jay and Timbre both begin by comparing the quantity discretely. While Shaun suggests they compare areas between curves to generate a comparison of quantities. Both of these are consistent with the maps created of their blends of quantity of water in the reservoir.

However, it is interesting that Shaun does not suggest comparing the points on the quantity graph discretely as Timbre does. Instead he suggests a new method of determining the quantity by comparing areas between the curves. This suggests that although Shaun was able to generate a correct graph of the quantity of water in the reservoir, (by interpreting the quantity of water as the antiderivative of the net flow), he did not have a fully connected understanding of that graph nearly 20 minutes after beginning the problem. In fact, only Timbre referred to her graph of quantity in order to compare the quantities at the beginning and ending of the year. The other students returned to the graphs of inflow and outflow to reason about the quantities. Therefore, the maps created along the way do not tell the whole story. We can not view the maps as a full representation of the students' understanding, they are merely analytical tools to help the researcher analyze and theorize about students emergent understandings.

Example 4: Jay Creates a New Graph of Quantity

About a minute after Shaun suggested they compare the areas between curves, Jay began to change his conception of quantity.

(00:23:52) **Jay**: Because since this is the just the rate of flow, um, it's just gonna be something like what's important is the whole graph because we can't really like, since we don't have a single number.

(00:24:04) **Shaun**: Um hmm

(00:24:06) **Jay**: we got, we're, we, like you said we're gonna be looking at the whole thing. If the outflow's been higher the entire time we can assume that the quantity has, is a total negative.

00:26:14) **Jay**: Well, you will definitely be getting a negative rate of flow, but it might not have allowed enough time for the quantity to go below this starting. So I think, just like, I guess our original way of going about this is not as effective as what we just saw. What he just told us.

Jay, Marcus, and Shaun began to compare areas between curves to determine the beginning and ending quantities of water in the reservoir. About 45 minutes into the task, Jay and Marcus re-created their graphs of quantity reasoning about the area between curves.

(00:51:42) **Jay**: …Okay, as far as my curvature went, I just kind of visualized in my head it's zero point you know where it's evened out is probably right before April so I, you know right before April, had it zero out so that would mean, you know, it's …It's overall quantity is just a little above April, or a little above the line.

(00:52:09) **Shaun**: um hum

(00:52:12) **Jay**: Towards July it's [the quantity is] increasing, but it's increasing, it's acceleration [second derivative], I don't, I hate saying that, but it's like rate of flow kind of thing, it's [second derivative is] going down and so the curve is going up [quantity] because it's [the quantity is] having less increase over time and so it's [the quantity is] having less and so it's [the quantity is] leveling out [at July].

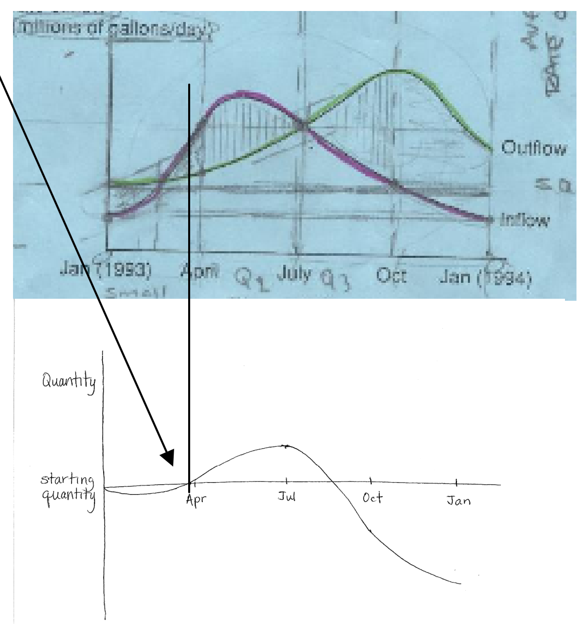


Figure 10: Jay's New Graph of Quantity

As Jay began to reason about the quantity using the area between curves, he created a new blend for the quantity of water in the reservoir (Figure 11). This new blend brings together his previous bowl metaphor with the context of the water to create a meaning for each of the bowls represented in the graph.

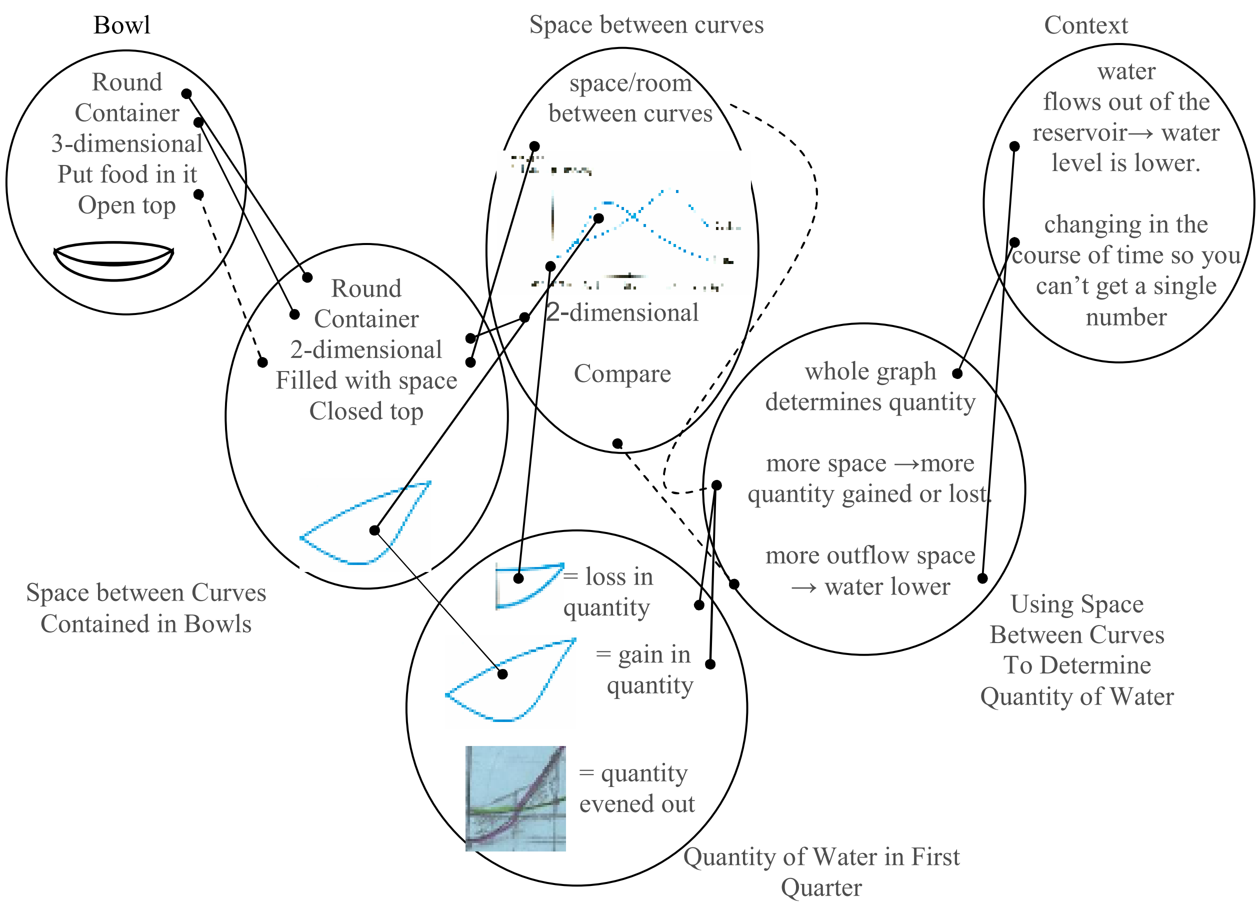


Figure 11: Jay's New Blend for Quantity of Water in the First Quarter

Mathematical Meaning Emerges

Whether representing meaning communicated over a period of time through group discussion as in example 1, or communicated in one utterance as in example 2, the exercise of creating a map of a blend helps to highlight, for the researcher, the concepts and ideas being built and discussed as well as the connections students are making between the concepts and ideas. We have found that it is usually not obvious what spaces are inputs, what ideas are being connected, how those connections are carried into the blend or even what the blended space should be called. It requires a detailed examination of the video, transcripts, and student work.

The act of creating the map requires the researcher to look at more than just word choices, gestures, or inscriptions. It requires the researcher to theorize about the meaning that is being created by the students and highlights the elements and connections that are being used by the students to create meaning. The map of the blend becomes a representation of the researcher's understanding of the student's sense making as well as a representation of the students' conceptual blends. A study of the data along with the maps of the blends helps the researcher to examine, create, re-examine, refine, and develop theories of students understanding of the mathematics. The mathematics does not get lost in the analysis, but remains a central focus throughout the process.

Results

Through sustained improvisational inquiry, students built a series of individual and collaborative conceptual blends for the quantity of water in the reservoir throughout the exploration. In particular, students called upon their constructed knowledge of and the connections between position, velocity, and acceleration, derivative as rate of change, the context of the reservoir, and everyday experience and language. With these blends, students began to make sense of concavity, area between curves, the first and second derivatives, and the antiderivative, as well as other intermediate calculus content important to the Fundamental Theorem of Calculus. Viewing the mathematical connections along with the construction of meaning through the lens of conceptual blending allowed us to more fully characterize the student's construction of meaning of important calculus ideas.

There are several strengths to conceptual blending as a lens for analysis. First, conceptual blending is very flexible in the unit of analysis and the specificity of the map. One can map conversations or single utterances. Maps of conceptual blends can be either general or very detailed helping to illuminate the development of meaning at many different levels. Second, the elements of the input and blended spaces and the connections between them are highlighted as maps of conceptual blends are constructed and analyzed along with the video, transcripts, and student work. After mapping the blends, the emergent connections students are making amongst their previous knowledge and the creative nature of these connections become an explicit part of the analysis. Third, the maps bring out emergent structure or the meaning that students are building for important mathematics. Finally, the maps aid comparisons between conceptions held by different students, or the same student over time, and help to illuminate the mathematics as it is built. The detailed analysis involved in creating blends organically over time provides insights into the meanings that are difficult to get otherwise. The maps help in checking current theories.

While the blends are a creation of the students, the maps are creations of the researcher. Some connections and spaces are merely implied in the discourse, leaving room for the researcher to misinterpret blends or emergent mathematical meanings. In addition adding researcher created artifacts to the data could introduce bias. Furthermore, while the lens of conceptual blending makes the meaning much more salient, it remains challenging to interpret students' meaning as they see it. Just because you can see the blend doesn't imply that you know what it means for the student. To mitigate these possible weaknesses it is, therefore, very important to use blends as a secondary source and continually go back to primary sources of data for confirmation of emergent theories.

Implications

Creating an environment where agency is central allows students to: build meaning for a task in an organic way rather than in a predetermined trajectory, draw on previous knowledge and experiences as they see fit to build meaningful conceptual blends create meaning for and, build connected understanding of important mathematics. Used as an analysis tool, conceptual blending is helpful in building a pathway for the researcher between Discourse and students connected understanding.

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[i] We see our model for learning as different from Simon's hypothetical learning trajectory (Simon & Tzur, 2004). In our case each task has the potential to lead students to certain calculus topics, but the students determine their own paths and their own content. For instance, if the students' inquiry did not lead to the concept of derivative on a particular day, that was not seen as a failure of the hypothetical learning trajectory. Instead the areas where their inquiry did take them were pursued in detail.

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